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PLANE-PARALLEL STATE OF A COMPOSITE WEDGE

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We consider the stressed state at the edge of the contact line of a composite wedge-shaped body made of materials which obey a power law under plane stress. This topic was investigated thoroughly in [1] for linearly elastic composite bodies and various aspects of low-stress composite bodies with power-law hardening of material under plane strain [2]. Similar studies were made in [3] for the edge of the contact surface of an inhomogeneous composite wedge under longitudinal shear and plane strain. Low stress concentration for a plane stressed state is clearly more complicated to study than is the corresponding problem of plane strain.

In our work the behavior of the stress field near the edge of the contact surface of a composite body, when the surfaces that form an edge are stress-free, is studied for the plane stressed state by means of a local solution (Fig. 1).

In the polar coordinate system a corner is chosen as the origin, the axis $\theta = 0$ is directed along the contact surface, and the z axis is perpendicular to the plane of the body. In each cross-sectional region we have the equilibrium equation

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0,$$
$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2}{r}\tau_{r\theta} = 0. \tag{1}$$

The hardening law is written as

$$\sigma_0 = k\varepsilon_0^m, \tag{2}$$

where

$$\sigma_0 = \frac{1}{\sqrt{3}}\sqrt{\sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2 + 3\tau_{r\theta}^2} \text{ and } \varepsilon_0 = 2\sqrt{\varepsilon_r^2 + \varepsilon_r\varepsilon_\theta + \varepsilon_\theta^2 + \gamma_{r\theta}^2}$$

respectively, are the intensities of the shearing stress and shearing strain, and m is the hardening index. The two materials have the same hardening index m but different bulk moduli k.

The relations between the components of the strains and the displacements are

$$\varepsilon_r = \frac{\partial u}{\partial r},\ \varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta},\ 2\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial\theta};$$

and the relations between the components of the stresses and strains are

$$\sigma_r - \sigma = 2k\varepsilon_0^{m-1}\varepsilon_r,\ \sigma_\theta - \sigma = 2k\varepsilon_0^{m-1}\varepsilon_\theta,$$
$$\sigma_z = 0,\ \tau_{r\theta} = 2k\varepsilon_0^{m-1}\gamma_{r\theta}. \tag{3}$$

Here $\sigma = 1/3(\sigma_r + \sigma_\theta)$ is the mean stress; we have also adopted the incompressibility condition $\varepsilon_r + \varepsilon_\theta + \varepsilon_z = 0$.

For each wedge-shaped region $-\beta \le \theta \le 0$ and $0 \le \theta \le \alpha$ we look for the displacement field in the form

$$u_i(r,\theta) = r^{1-\lambda}f_i(\theta,\lambda),\ v_i(r,\theta) = r^{1-\lambda}\psi_i(\theta,\lambda), \tag{4}$$

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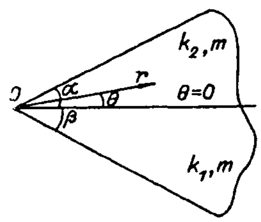

Fig. 1

where $f_i(\theta, \lambda)$ and $\psi_i(\theta, \lambda)$ are arbitrary functions and λ is an arbitrary constant. The quantities in these regions are denoted by the indices $i = 1, 2$, respectively.

Using representations (3) and (4), we write the components of the stress in terms of the unknown functions $f_i(\theta, \lambda)$ and $\psi_i(\theta, \lambda)$:

$$\begin{gathered}\sigma_{ri} = 2k_i r^{-\lambda m}(\psi_i' + (3 - 2\lambda)f_i)\chi_i,\\ \sigma_{\theta i} = 2k_i r^{-\lambda m}(2\psi_i' + (3 - \lambda)f_i)\chi_i,\\ \tau_{r\theta i} = k_i r^{-\lambda m}(f_i' - \lambda\psi_i)\chi_i,\\ \chi_i = (4(1 - \lambda)^2 f_i^2 + 4(1 - \lambda)f_i(f_i + \psi_i') + 4(f_i + \psi_i')^2 + (f_i' - \lambda\psi_i)^2)^{(m-1)/2}.\end{gathered} \tag{5}$$

The functions $f_i(\theta, \lambda)$ and $\psi_i(\theta, \lambda)$ in each region satisfy the system of ordinary differential equations

$$\begin{gathered}((f_i' - \lambda\psi_i)\chi_i)' - 2((1 + \lambda m)\psi_i' + \lambda(1 + m(3 - 2\lambda))f_i)\chi_i = 0,\\ ((2\psi_i' + (3 - \lambda)f_i)\chi_i)' + \left(1 - \frac{1}{2}\lambda m\right)(f_i' - \lambda\psi_i)\chi_i = 0.\end{gathered} \tag{6}$$

In the absence of external forces near the corner we have the boundary conditions

$$\begin{gathered}2\psi_1' + (3 - \lambda)f_1 = 0,\ 2\psi_2' + (3 - \lambda)f_2 = 0,\\ f_1' - \lambda\psi_1 = 0 \text{ for } \theta = -\beta,\ f_2' - \lambda\psi_2 = 0 \text{ for } \theta = \alpha.\end{gathered} \tag{7}$$

At the contact surface $\theta = 0$ we have the conditions for equal stresses $\sigma_{\theta i}$ and $\tau_{r\theta i}$ and for continuity of the displacements u_i and v_i, i.e., the conditions

$$\begin{gathered}(2\psi_1' + (3 - \lambda)f_1)\chi_1 = \gamma(2\psi_2' + (3 - \lambda)f_2)\chi_2,\\ (f_1' - \lambda\psi_1)\chi_1 = \gamma(f_2' - \lambda\psi_2)\chi_2,\ f_1 = f_2,\ \psi_1 = \psi_2,\ \gamma = \frac{k_2}{k_1} \text{ for } \theta = 0.\end{gathered} \tag{8}$$

The system of differential equations (6) with the boundary conditions (7) and (8) in principle determines the functions $f_i(\theta, \lambda)$ and $\psi_i(\theta, \lambda)$ to within a constant indeterminate multiplier in the regions $-\beta \le \theta \le 0$ and $0 \le \theta \le \alpha$ and the value of λ as a function of the parameters α, β, γ, and m. Assigning different numerical values to λ, we find the relations between α, β, γ, and m from (6)-(8) by a numerical method. The condition $\lambda < 0$ in the space of those parameters determines the zone of low stress and the condition $\lambda > 0$, the zone of high stress concentration. Assuming that $\lambda = \lambda_* > 0$, we can determine the family of curves of identical degrees of stress concentration from (6)-(8) by numerical methods. These curves $\beta = \beta(\alpha, \gamma, m)$ are traces of the hypersurface $\lambda(\alpha, \beta, \gamma, m) = \lambda_*$ in the α, β coordinate plane as a function of γ and m.

A special examination must be made of Eqs. (1)-(3) if the stresses are finite, i.e., when $\lambda = 0$. In this case we look for the displacement components that satisfy the incompressibility condition for materials in the form

$$u_i(r,\theta) = Crf_i(\theta),\ v_i(r,\theta) = Cr\psi_i(\theta) + Cr\ln r. \tag{9}$$

Here $f_i(\theta)$ and $\psi_i(\theta)$ are arbitrary functions and C is an arbitrary constant.

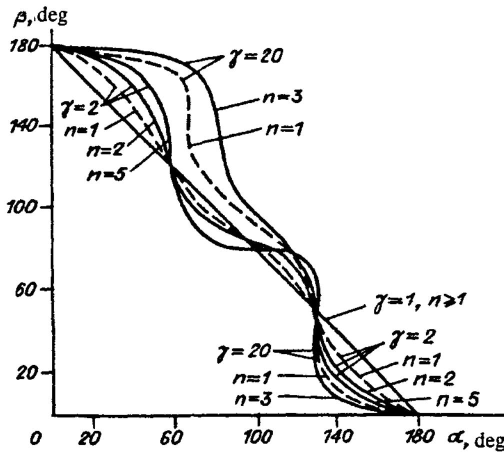

Fig. 2

Proceeding from the power law of hardening, we write the stress components as

$$\begin{gathered}\sigma_{ri} = 2k_iB(\psi_i' + 3f_i)\chi_i,\\ \sigma_{\theta i} = 2k_iB(2\psi_i' + 3f_i)\chi_i,\ \tau_{r\theta i} = k_iB(f_i' + 1)\chi_i,\\ \chi_i = ((f_i' + 1)^2 + 4(3f_i^2 + 3f_i\psi_i' + \psi_i'^2))^{(m-1)/2},\ B = C\mid C\mid^{m-1}.\end{gathered} \tag{10}$$

On substituting the stress equation (10) into the equilibrium equation (1), we obtain a system of equations for determining the unknown functions $f_i(\theta)$ and $\psi_i(\theta)$:

$$((f_i' + 1)\chi_i)' - 2\psi_i'\chi_i = 0,\ ((2\psi_i' + 3f_i)\chi_i)' + (f_i' + 1)\chi_i = 0. \tag{11}$$

Assuming the condition for zero normal and shear stresses at the external edges of the composite wedge, we have the boundary conditions

$$\begin{gathered}2\psi_1' + 3f_1 = 0,\ 2\psi_2' + 3f_2 = 0,\\ f_1' = -1 \text{ for } \theta = -\beta,\ f_2' = -1 \text{ for } \theta = \alpha\end{gathered} \tag{12}$$

and the contact-surface conditions

$$\begin{gathered}(2\psi_1' + 3f_1)\chi_1 = \gamma(2\psi_2' + 3f_2)\chi_2,\ (f_1' + 1)\chi_1 = \gamma(f_2' + 1)\chi_2,\\ f_1 = f_2,\ \psi_1 = \psi_2,\ \gamma = \frac{k_2}{k_1} \text{ for } \theta = 0.\end{gathered} \tag{13}$$

The system of differential equations (11) with boundary and contact conditions (12) and (13) in principle determines the hypersurface $\Phi(\alpha, \beta, \gamma, m) = 0$ of infinite stresses, which separates the zone of low stress concentration from the zone of high stress concentration. To construct this surface it is appropriate to reduce the system of differential equations (11) to a system of eight first-order differential equations, which is more amenable to numerical integration:

$$\begin{gathered}\psi_i' = \frac{1}{2}(F_i - 3f_i),\ f_i' = \tau_i,\ F_i' = \Phi_i,\\ \tau_i' = \frac{I_i(F_i - 3f_i) - (m-1)(\tau_i + 1)(F_i\Phi_i + 3f_i\tau_i)}{I_i + (m-1)(\tau_i + 1)^2},\ i = 1,2.\end{gathered} \tag{14}$$

Here

$$I_i = (\tau_i + 1)^2 + F_i^2 + 3f_i^2;$$
$$\Phi_i = \{(1 - m)(\tau_i + 1)F_i[3(m - 1)(\tau_i + 1)f_i\tau_i - I_i(F_i - 3f_i)]$$
$$+ [I_i(\tau_i + 1) + 3(m - 1)f_i\tau_iF_i][I_i + (m - 1)(\tau_i + 1)^2]\}/$$
$$\{(1 - m)^2(\tau_i + 1)^2F_i^2 - [I_i + (m - 1)F_i^2][I_i + (m - 1)(\tau_i + 1)^2]\}.$$

The boundary conditions for the system (14) are

$$F_1(\theta) = 0,\ \tau_1(\theta) = -1 \text{ for } \theta = -\beta,$$
$$F_2(\theta) = 0,\ \tau_2(\theta) = -1 \text{ for } \theta = \alpha, \tag{15}$$

and the contact conditions (13) become

$$F_1\chi_1 = \gamma F_2\chi_2,\ (\tau_1 + 1)\chi_1 = \gamma(\tau_2 + 1)\chi_2,$$
$$f_1 = f_2,\ \psi_1 = \psi_2,\ \gamma = \frac{k_2}{k_1} \text{ for } \theta = 0. \tag{16}$$

The stress components (10) is written as

$$\sigma_{ri} = k_iB(F_i + 3f_i)\chi_i,\ \sigma_{\theta i} = 2k_iBF_i\chi_i,$$
$$\tau_{r\theta i} = k_iB(\tau_i + 1)\chi_i,\ \chi_i = [(\tau_i + 1)^2 + 3f_i^2 + F_i^2]^{(m-1)/2}. \tag{17}$$

When the wedge is made of a single homogeneous material, i.e., for $\gamma = 1$, the system of equations (14) holds only for the region $0 \le \theta \le \alpha$ and the boundary conditions (15) is written as

$$F(\theta) = 0,\ F(\theta) = 0,$$
$$\tau(\theta) = -1 \text{ for } \theta = 0,\ \tau(\theta) = -1 \text{ for } \theta = \alpha. \tag{18}$$

The system of equations (14) with the conditions (18) establish a relation between α and m.

If the composite wedge is made of linearly elastic materials, then on assuming that m = 1 in Eqs. (12) we have the equation $f_i''' = 4f_i' + 1 = 0$, with the general solution

$$f_i(\theta) = C_{1i} + C_{2i}\sin 2\theta + C_{3i}\cos 2\theta - \frac{\theta}{4}, \tag{19}$$

where C_{1i}, C_{2i} and C_{3i} are arbitrary constants. When we use the boundary and contact conditions (15), (16) for m = 1 and the expression $f_i(\theta)$, then from (19) we obtain

$$\begin{aligned}
&[3\sin 2\alpha + 3(\gamma - 1)\sin 2\alpha \cdot \cos 2\beta + 3\sin 2\beta \\
&+ \cos 2\alpha \cdot \sin 2\beta + (2\alpha - 3(\gamma - 1))\sin 2\alpha \cdot \sin 2\beta] \\
&\times [\gamma\sin 2\alpha \cdot \cos 2\beta \cdot (1 + 3\gamma - \cos 2\beta) + 4\gamma\cos 2\alpha \cdot \sin 2\beta \\
&- \gamma\sin 2\alpha \cdot \sin^2 2\beta - 3\gamma^2\sin 2\alpha \cdot \sin 2\beta] \\
&- [3(\gamma\sin 2\alpha \cdot \sin 2\beta - \gamma\sin 2\alpha \cdot \cos 2\beta - \cos 2\alpha \cdot \sin 2\beta) \\
&- \sin 2\beta][(3\gamma(\gamma - 1) + 2\beta + (\gamma - 1)\sin 2\beta)\sin 2\alpha \cdot \sin 2\beta \\
&- (3\gamma + 1)\sin 2\alpha - (3\gamma^2 - 2\gamma - 2 - (\gamma - 1)\cos 2\beta)\sin 2\alpha \cdot \cos 2\beta - 4\gamma\sin 2\beta] = 0.
\end{aligned} \tag{20}$$

Equation (20) in the α, β plane defines a family of limiting curves, which separate the zone of low stress concentration from the zone of strong concentration for linearly incompressible materials in the case of a plane stressed state.

Figure 2 shows a family of limiting curves $\beta = \beta(\alpha, \gamma, m)$, drawn with the aid of numerical solutions of the system of equations (14) for the boundary conditions (15), (16); these curves separate the zone of low stress concentration (below the curves) from the zone of strong concentration (above the curves). The dashed lines indicate such curves for a linearly elastic incompressible material. The parameter $n = 1/m$ is given on the graphs.

The system (14) with the conditions (15), (16) is integrated numerically by the following method. If we eliminate the functions ψ_i, which do not appear in the other equations, we obtain a system of sixth-order differential equations with seven boundary conditions. From the seventh additional condition we determine the relation between α and β.

For each fixed α and β system (14) with six boundary conditions is solved by the shooting method, the gist of which is as follows. Assuming that $f_2(\alpha) = p$, we can solve the system by using a method of solving the Cauchy problem (in our problem we use the fourth-order Runge-Kutta method). System (14) is solved if the value for which $F_1(-\beta)$ is zero [$G(p) = 0$] is found. A functional relation is obtained between randomly chosen p and $F_1(-\beta)$, $G: p \to F_1(-\beta)$. The function G(p) is not known in explicit form, but its value can be calculated for any p by numerical integration of system (14). For each α, by varying β discretely with a constant step starting from $\beta = 0$ we find the first (smallest) value of β for which the solution obtained for the system satisfies the additional seventh condition $\tau_1(-\beta) = -1$.

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